Solutions to Problem Set 3

1. The critical density is $3H_0^2/8\pi G$. But, $H_0 = 2.13h \times 10^{-34}$ eV, and $G = 1/m_{\rm pl}^2 = 1/(1.22 \times 10^{28}{\rm eV})^2$, so $\rho_c = 8.1h^2 \times 10^{-11}{\rm eV}^4$. The ratio

$$\frac{\rho_\Lambda}{3H^2/(8\pi G)} = (\rho_\Lambda/\rho_c)_0 \left(\frac{H_0}{H}\right)^2$$

where subscript 0 means evaluate today, where it is assumed to be 0.7. Again, by assumption, the universe is forever radiation dominated (clearly not true today, but a good approximation early on), so $H/H_0 = a^{-2}$. The temperature also scales as a^{-1}, so $H/H_0 = (T/T_0)^2$ with $T_0 = 2.7{\rm K} = 2.3 \times 10^{-4}$ eV. So,

$$\frac{\rho_\Lambda}{3H^2/(8\pi G)} = 0.7 \left(\frac{T_0}{T}\right)^4.$$

At the Planck scale, $T_0/T = 2.3 \times 10^{-4}/1.22 \times 10^{28}$, so

$$\frac{\rho_\Lambda}{3H^2/(8\pi G)} = 9 \times 10^{-128}.$$

This is the so-called fine-tuning problem: For the cosmological constant to be important today, it had to have been fine-tuned to an absurdly small value at early times. It's a deep problem.

2. We need to do the integral

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{da}{a}\left[\Omega_\Lambda + \frac{1-\Omega_\Lambda}{a^3}\right]^{-1/2}$$

for $\Omega_\Lambda = 0.7$ and 0. The latter case can be done analytically:

$$\int_0^1 \frac{da}{a} a^{3/2} = \frac{2}{3}.$$

So $t_0 = 2/3H_0 = 0.67 \times 10^{10} h^{-1}$ yrs. When Ω_Λ is not zero, the integral needs to be done numerically. I find

$$\int_0^1 \frac{da}{a}\left[0.7 + \frac{0.3}{a^3}\right]^{-1/2} = 0.96.$$

So for fixed Hubble constant, a cosmological constant universe is older than a matter dominated one, older by a factor of $.96/.67 = 1.43$. For $h = 0.7$, a cosmological constant universe has an age of 14 billion years, in accord with other observations of the age of the universe.

3. (a) To get from Kelvin to eV, use $k_B = \mathrm{eV}/(11605K)$. So 2.728K $\to k_B 2.728\mathrm{K} =$ (2.728/11605) eV. Or 2.35×10^{-4} eV.

(b) Since $T = 2.35 \times 10^{-4}$ eV,

$$\rho_\gamma = \frac{\pi^2 T^4}{15} = 2 \times 10^{-15} \mathrm{eV}^4.$$

To get this in g cm^{-3}, first divide by $(\hbar c)^3 = (1.97 \times 10^{-5}\mathrm{eV\ cm})^3$ to get $0.2625\mathrm{eV\ cm}^{-3}$. Then to change from eV to grams, remember that the mass of the proton is either 1.67×10^{-24} g or 0.938×10^9 eV, so 1 eV $= 1.78 \times 10^{-33}$ g. Therefore, $\rho_\gamma = 4.67 \times 10^{-34}$ g cm^{-3}.

(c) We have parametrized $H_0 = 100h\,\mathrm{km\ sec}^{-1}\ \mathrm{Mpc}^{-1}$, or using the fact that one Mpc is equal to $3.1 \times 10^{19}\,\mathrm{km}$, $H_0 = 3.23h \times 10^{-18}\ \mathrm{sec}^{-1}$. To get this into inverse cm, divide by the speed of light, $c = 3 \times 10^{10}$ cm sec $^{-1}$; then $H_0 = 1.1h \times 10^{-28}$ cm. Or $H_0^{-1} = 9.3h^{-1} \times 10^{27}$ cm.

(d) To get the Planck mass (1.2×10^{28} eV) into degrees Kelvin, multiply by $k_B^{-1} =$ 11605K/eV; then $m_{\mathrm{Pl}} = 1.4 \times 10^{32}\,\mathrm{K}$. To get it into inverse cm, divide by $\hbar c = 1.97 \times 10^{-5}$ eV cm to get $m_{\mathrm{Pl}} = 6.1 \times 10^{32}\ \mathrm{cm}^{-1}$. To get this is units of time, multiply by the speed of light to get $m_{\mathrm{Pl}} = 6.1 \times 10^{32} \times 3 \times 10^{10}$ cm sec^{-1}, or $m_{\mathrm{Pl}} = 1.8 \times 10^{43}\ \mathrm{sec}^{-1}$.

4.

$$\vec{A} \cdot \vec{B} = A_i B_i$$

$$\vec{A} \cdot \vec{B}\vec{C} \cdot \vec{D} = A_i B_i C_j D_j$$

I.e. distinuish the dummy index in one dot product from the one in the other.

$$\vec{\nabla} f = \frac{\partial f}{\partial x_i}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_i}{\partial x_i}$$